Geometric Algebra: It's not Algebraic Geometry!

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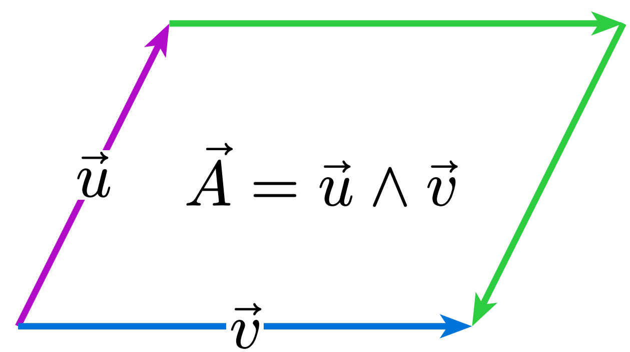
Geometric algebra...

- Makes things intuitive
- Generalizes “weird” phenomena
  - Cross product
  - Quaternions for rotation
  - A lot of physics
- Provides a more powerful framework than linear algebra
  - You can add scalars and vectors (sort of)!

Geometric Primitives

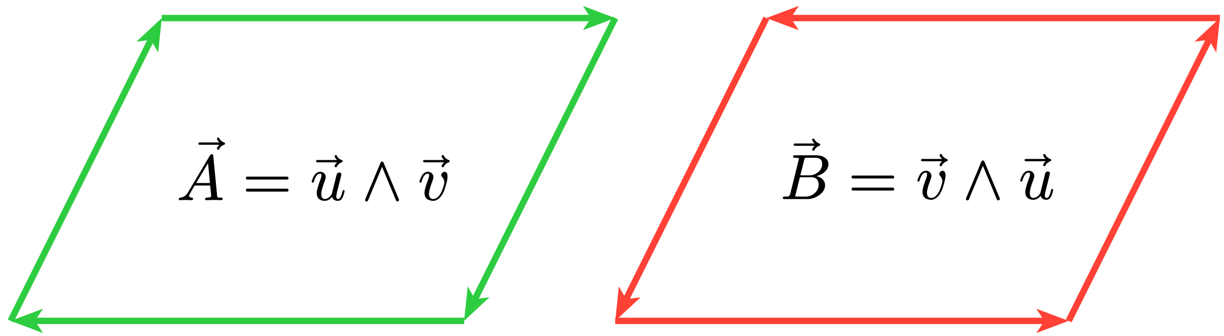
For now, consider 2D space. Let $\vec{u} = (1, 2)^T$ and $\vec{v} = (3, 0)^T$:

Then, we define their **wedge product**, $\vec{A} := \vec{u} \wedge \vec{v}$:

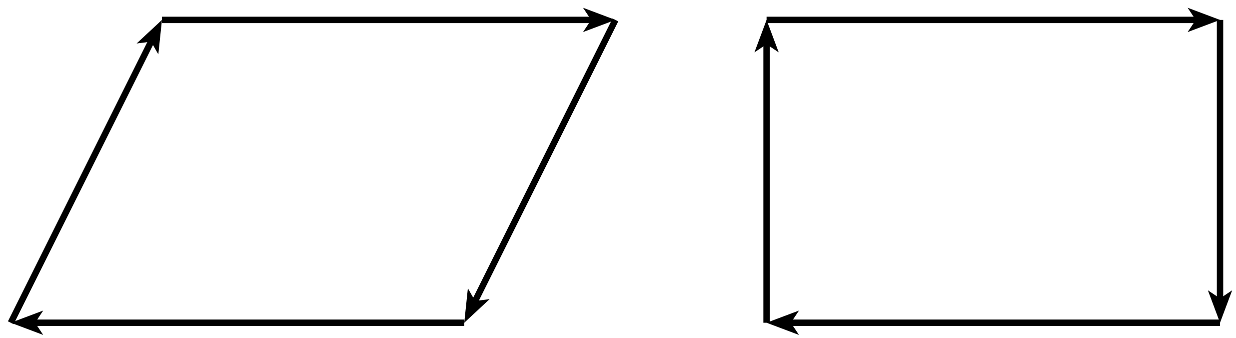


as the oriented 2D area formed by joining the two vectors. We call $\vec{A}$ a **bivector**. Note that $\vec{A}$ has an area and an orientation.

The wedge product is **anti-commutative**, i.e. $\vec{u} \wedge \vec{v} = -\vec{v} \wedge \vec{u}$. This can be visualized as the orientation of the two bivectors these wedge products form:



In fact, the wedge product is uniquely determined by the magnitude and orientation. The following two bivectors are equivalent:



In other words, shape does not matter.

More Intuition for Wedge Product

We can infer two more properties about the wedge product:

1. The wedge product is **bilinear**.
2. For parallel $\vec{u}, \vec{v}$, we have that $\vec{u} \wedge \vec{v} = 0$.

Hence, let e_1, e_2 be the standard basis vectors of $\mathbb{R}^2$. We can express the wedge product of $\vec{u}, \vec{v} \in \mathbb{R}^2$ in terms of these basis vectors:

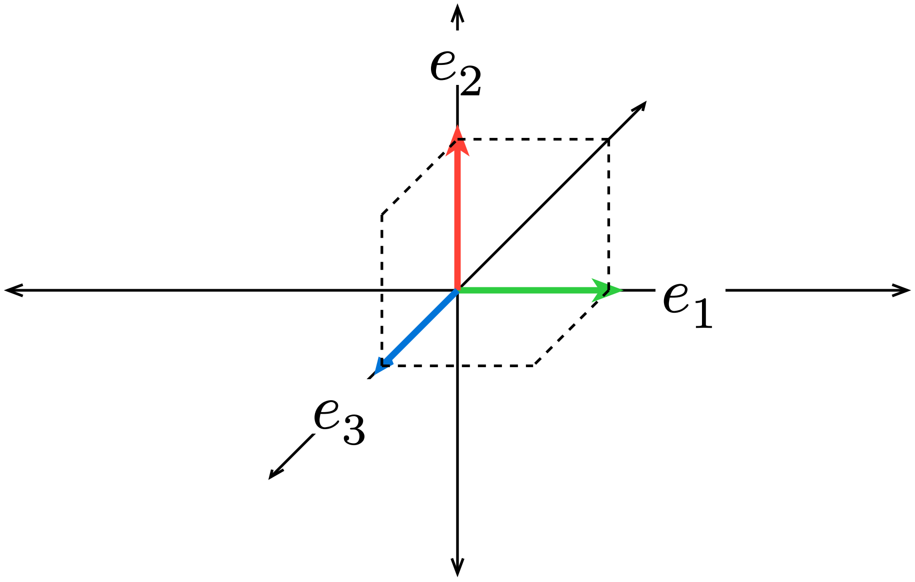
$$\begin{aligned}
\vec{u} \wedge \vec{v} &= (\vec{u}_1 e_1 + \vec{u}_2 e_2) \wedge (\vec{v}_1 e_1 + \vec{v}_2 e_2) \\
&= \vec{u}_1 \vec{v}_2 (e_1 \wedge e_2) + \vec{u}_2 \vec{v}_1 (e_2 \wedge e_1) \\
&= \vec{u}_1 \vec{v}_2 (e_1 \wedge e_2) - \vec{u}_2 \vec{v}_1 (e_1 \wedge e_2) \\
&= (\vec{u}_1 \vec{v}_2 - \vec{u}_2 \vec{v}_1)(e_1 \wedge e_2)
\end{aligned}$$

Since $e_1 \wedge e_2$ forms the unit square, we can see that the area of $\vec{u} \wedge \vec{v}$ equals $B_{12} := \vec{u}_1 \vec{v}_2 - \vec{u}_2 \vec{v}_1 = \sin(\theta)\|u\|\|v\|$. Furthermore, $\vec{u} \wedge \vec{v}$ form the unit square scaled by B_{12}.

2D $\to$ 3D

For now, bivectors are analogous to planes. In 3D space, bivectors have three components, instead of one. Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$.

Vectors and bivectors in 3D:



$$\vec{v} = v_1 e_1 + v_2 e_2 + v_3 e_3$$

$$\vec{B} = B_{12}(e_1 \wedge e_2) + B_{13}(e_1 \wedge e_3) + B_{23}(e_2 \wedge e_3)$$

For a 3D bivector, $\vec{B} = B_{12}(e_1 \wedge e_2) + B_{13}(e_1 \wedge e_3) + B_{23}(e_2 \wedge e_3)$, the components B_{ij} are the projections of $\vec{B}$ onto the basis planes:

$$B_{12} = \vec{u}_1\vec{v}_2 - \vec{u}_2\vec{v}_1$$
$$B_{13} = \vec{u}_1\vec{v}_3 - \vec{u}_3\vec{v}_1$$
$$B_{23} = \vec{u}_2\vec{v}_3 - \vec{u}_3\vec{v}_2$$

Wait, you look familiar...

$$\begin{aligned}\vec{u} \wedge \vec{v} = \ & (\vec{u}_1\vec{v}_2 - \vec{u}_2\vec{v}_1)(e_1 \wedge e_2) \\ & +(\vec{u}_1\vec{v}_3 - \vec{u}_3\vec{v}_1)(e_1 \wedge e_3) \\ & +(\vec{u}_2\vec{v}_3 - \vec{u}_3\vec{v}_2)(e_2 \wedge e_3)\end{aligned}$$

$$\begin{aligned}\vec{u} \times \vec{v} = \ & (\vec{u}_1\vec{v}_2 - \vec{u}_2\vec{v}_1)e_3 \\ & -(\vec{u}_1\vec{v}_3 - \vec{u}_3\vec{v}_1)e_2 \\ & +(\vec{u}_2\vec{v}_3 - \vec{u}_3\vec{v}_2)e_1\end{aligned}$$

Now introducing...

The Geometric Product

The Geometric Product

At the heart of geometric algebra is the geometric product:

$$\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v}$$

Similar to how an imaginary number has a real and imaginary part, the geometric product is a **multivector**; it has a scalar and bivector part.

Let $\vec{u}$ be a vector, and let e_i, e_j be distinct basis vectors:

$$\vec{u}\vec{u} = \vec{u} \cdot \vec{u} + \vec{u} \wedge \vec{u} = \|\vec{u}\|^2$$
$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j = e_i \wedge e_j = -e_j e_i$$

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$. Explicitly, their geometric product is:

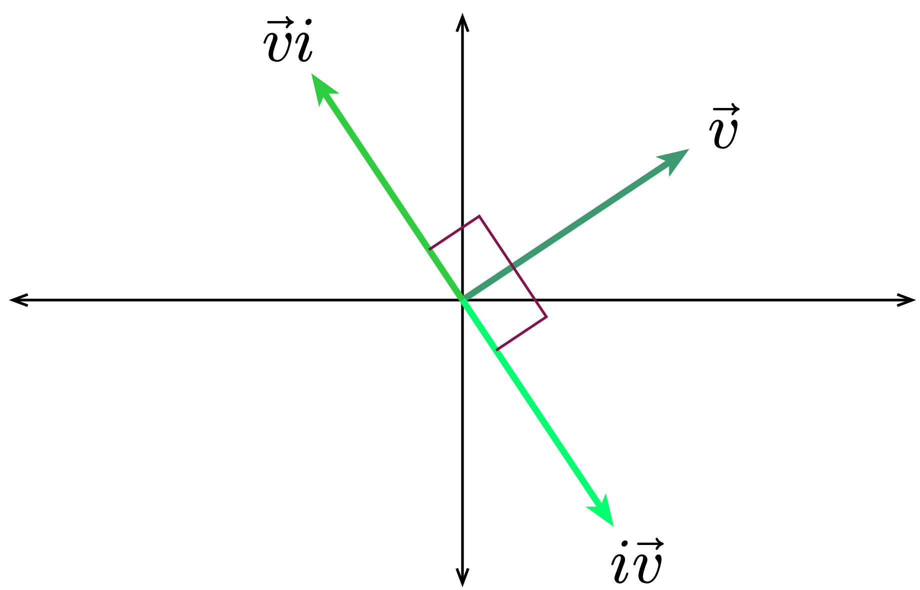
$$\begin{aligned} \vec{u}\vec{v} = \; & \vec{u}_1\vec{v}_1 + \vec{u}_2\vec{v}_2 + \vec{u}_3\vec{v}_3 + (\vec{u}_1\vec{v}_2 - \vec{u}_2\vec{v}_1)(e_1 \wedge e_2) \\ & + (\vec{u}_1\vec{v}_3 - \vec{u}_3\vec{v}_1)(e_1 \wedge e_3) + (\vec{u}_2\vec{v}_3 - \vec{u}_3\vec{v}_2)(e_2 \wedge e_3) \end{aligned}$$

Imaginary Numbers?

We return to $\mathbb{R}^2$. Let e_1, e_2 be the basis vectors, and $i := e_1 e_2$.

Let $\vec{v} = ae_1 + be_2$. Then,

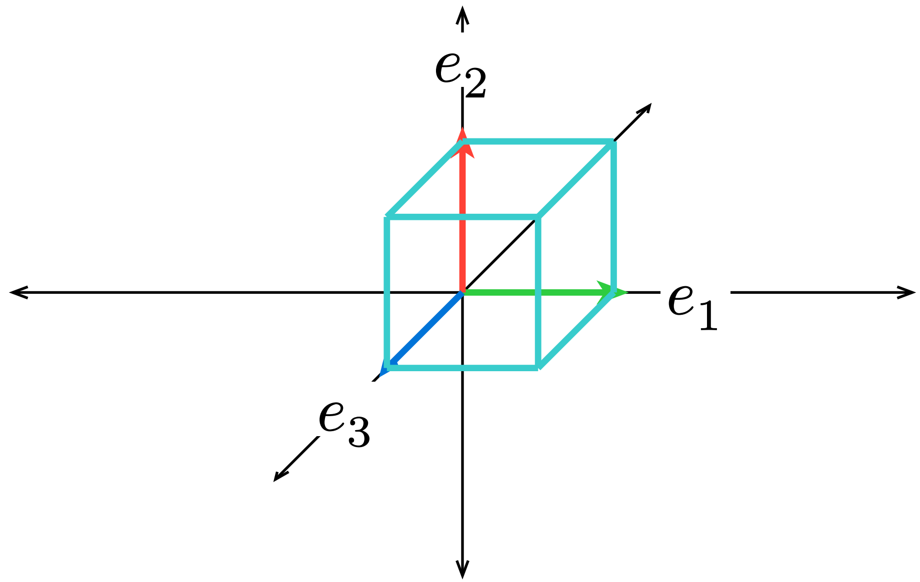
$$\vec{v}i = ae_2 - be_1, i\vec{v} = -ae_2 + be_1$$



Note that $i^2 = (e_1 e_2)^2 = -e_2 e_1 e_1 e_2 = -1$, just like imaginary numbers!

Imaginary Numbers?

In $\mathbb{R}^3$, we have the orthonormal basis e_1, e_2, e_3, and $i := e_1 e_2 e_3$. This value is a trivector, and it looks the unit cube (with orientation).



Here, i commutes with any multivector $\vec{A}$. Note that $e_1 i = e_2 e_3$ and $e_1 = -e_2 e_3 i$.

Remarkably, $\vec{u} \wedge \vec{v} = i(\vec{u} \times \vec{v})$.

Quaternions?

If $\boldsymbol{p}, \boldsymbol{q}$ are unit quaternions, then rotation of $\vec{v}$ by $\boldsymbol{pq}$ is $\boldsymbol{pq}\vec{v}(\boldsymbol{pq})^{-1}$. This looks a lot like rotation using rotors, but rotors are easier to understand.

Let $x = e_1e_2, y = e_1e_3, z = e_2e_3$:

	x	y	z
x	-1	z	$-y$
y	$-z$	-1	x
z	y	$-x$	-1

	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

$$(e_1e_2)^2 = (e_1e_3)^2 = (e_2e_3)^2 = (e_1e_2)(e_1e_3)(e_2e_3) = -1$$

$$i^2 = j^2 = k^2 = ijk = -1$$

Quaternions?

In computer graphics, the geometric product can be used to encode 3D rotations. Let $\vec{v}, \vec{a}, \vec{b}$ be linearly independent vectors. The reflection of $\vec{v}$ about $\vec{a}$ is given by:

$$\begin{aligned} R_{\vec{a}}(\vec{v}) &= \vec{v} - 2(\vec{v} \cdot \vec{a})\vec{a} \\ &= \vec{v} - 2\left(\frac{1}{2}(\vec{v}\vec{a} + \vec{a}\vec{v})\right)\vec{a} \\ &= \vec{v} - \vec{v}\vec{a}^2 - \vec{a}\vec{v}\vec{a} \\ &= -\vec{a}\vec{v}\vec{a} \end{aligned}$$

Likewise, reflecting about $\vec{a}$ then about $\vec{b}$ is given by:

$$R_{\vec{b}}(R_{\vec{a}}(\vec{v})) = -\vec{b}(-\vec{a}\vec{v}\vec{a})\vec{b} = \vec{ba}\vec{v}\vec{ab}$$

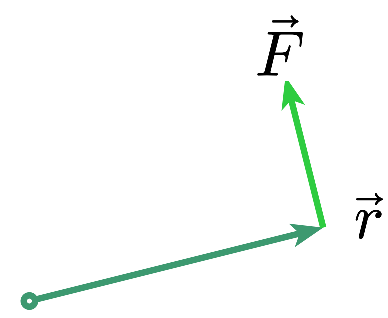
This encodes the rotation in the plane formed by $\vec{a}, \vec{b}$ by $2\angle\vec{a}\vec{b}$. The product $\vec{ab}$ is called a **rotor**.

Physical Applications

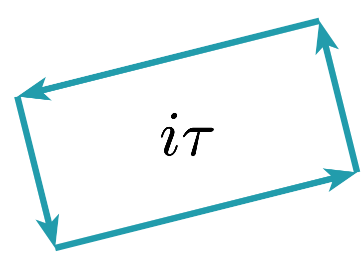
Physics Made Intuitive

In 2D, a bivector is a **pseudoscalar**. Likewise, trivectors are 3D pseudoscalars. This is because the basis of bivectors in 2D has one element, and the basis of trivectors in 3D have one element.

For example, consider torque, $\tau = \vec{r} \times \vec{F}$:



For some reason, τ points out of the page. But consider $i\tau$:



Which makes a lot more sense, in my opinion.

Maxwell's Equations

Recall Maxwell's equations, which describe electromagnetism.

	$\vec{E}$	$\vec{B}$
$\cdot$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$	$\vec{\nabla} \cdot \vec{B} = 0$
$\times$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

However, with geometric algebra, this can be made a lot simpler.

Maxwell's Equations

We can add scalars and vectors now, so let's define a new gradient and some new variables.

$$\begin{aligned} \nabla &= \frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla} \\ &= \frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \end{aligned}$$

$$J = c\rho - \vec{J}$$

$$F = \vec{E} + i\vec{B}$$

This gives us a new statement of Maxwell's equations...

$$\nabla F = J$$

Generalizing to Higher Dimensions

What happens when we go above 3D?

What is the geometric product of two bivectors?

$$(\text{Bivector})(\text{Bivector}) = \text{Scalar} + \text{Bivector} + \text{4-vector}$$

This breaks our $\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v}$ rule! Turns out, most of the time $\vec{u}\vec{v} \neq \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v}$. In general:

$$\begin{aligned}(r\text{-vector})(s\text{-vector}) = \; & (|r-s|)\text{-vector} \\ & +(|r-s|+2)\text{-vector} \\ & +(|r-s|+4)\text{-vector} \\ & \;\vdots \\ & +(|r+s|)\text{-vector}\end{aligned}$$

This gives us the general form for the geometric product, between any vectors in any dimension.

Let's reflect...

Let's reflect...

- In geometric algebra, you think about *oriented magnitudes*
- A lot of physical phenomena are pseudoscalars
- I lied to you guys in the abstract

Let V be a vector space with a symmetric bilinear form $B : V \times V \to \mathbb{K}$, where $\mathbb{K}$ is a field. Say $\otimes$ is the tensor product of $u, v \in V$, and consider the ideal $\mathcal{J}(V, B)$ generated by elements of the form $v \otimes w + w \otimes v - 2B(v, w)1$. Then, consider $T(V) := \bigoplus_{k \in \mathbb{Z}} T^k(V)$, where $T^k(V) = V \otimes V ... \otimes V$ is the k-fold tensor product. Naturally, the Clifford (Geometric) algebra of V is the quotient:

$$\mathrm{Cl}(V, B) = T(V)/\mathcal{J}(V, B)$$

Note that this is similar to the exterior algebra, which is $T(V)/\mathcal{I}(V, B)$, where $\mathcal{I}(V, B)$ is the ideal generated by elements of the form $v \otimes w + w \otimes v$.

Thanks to the...

MEF

Mathematics Endowment Fund

for funding my broke a.**